Some Results for Resource Games

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Abstract

Resource games, a branch of differentiable games, provide natural resources and environmental issues with useful insight into the human dimension. This paper consists of three results for resource games, which are mutually independent. The first topic is on the dynamic property of stationary points when a Markov perfect Nash equilibrium strategy of each player is differentiable at the points. The second one is concerned with a special Markovian strategy, the most rapid extinction. It is examined under what conditions the strategy constitutes a Markov perfect Nash equilibrium when the resource has externality. As the last topic, we propose a remedy against the "tragedy of commons". Each section corresponds to the three topics, respectively. The models commonly include just one state variable, the resource stock.

1 On stationary points of a Markov perfect Nash equilibrium

Markov perfect Nash equilibria are concentratedly considered in literature of economics. There are at least two reasons why economists favor this equilibrium concept. The first is that it is a straightforward translation into differential games of a very familiar equilibrium concept for a game in extensive form, subgame perfectness. The second is that it restricts the strategy space by assuming that players react only to the current state and this restriction opens availability of some mathematical techniques, so called, recursive methods, which are also familiar to economists. Due to the restriction of strategy space, the analysis becomes rather tractable. Nevertheless, several difficulties arise in the analysis. For example, there might be a continuum of equilibria and for an equilibrium, the equilibrium strategy might not be continuous nor have convenient convex properties. Reflecting such analytical difficulties, there are still few to be known on Markov perfect Nash equilibria. This section investigates what we can know if the equilibrium strategy has a property that it is differentiable at the stationary states.

1.1 Notation and definition

- $t \in [0, \infty)$ the continuous time.
- $N = \{1, 2, \dots, n\}$ the set of players. The set is finite ($2 \leq n < \infty$).
- $u_i : R_+ \to \bar{R}$ the instantaneous utility function of player $i \in N$, where R_+ is the set of non negative real numbers and $\bar{R}$ is the set of extended real numbers.
- $\rho_i > 0$ the discount rate of player $i \in N$.
- $X = R_+$ the state space.
- $f : X \to R$ the natural growth function for the resource. $f(0) = 0$.
- $(N, \{u_i, \rho_i\}_{i=1}^n, X, f)$ a resource game. For a special case such that for all $i \in N$, $u_i = u, \rho_i = \rho$, we denote the resource game by (N, u, ρ, X, f) and call it a symmetric resource game.
- $s_i : X \to R_+$ a stationary Markovian strategy of player $i \in N$.
- $S = \{s_1, s_2, \dots, s_n\}$ a strategy profile.
- For a strategy profile $S = \{s_1, s_2, \dots, s_n\}$ and a strategy s'_i, we adopt the conventional notation, $(s'_i, S_{-i}) = (s_1, \dots, s_{i-1}, s'_i, s_{i+1}, \dots, s_n)$.

Definition 1 *A strategy profile S is admissible if the following conditions are satisfied:*

For each initial condition $x \in X$,

1. *there exists an absolutely continuous function $x(t)$ that is the unique solution for the differential equation $\dot{x}(t) = f[x(t)] - \sum_{i=1}^N s_i[x(t)]$ starting from $x(0) = x$, and*
2. *For each $i \in N$, $c_i(t) = s_i[x(t)]$ is measurable.*

Remark 1 *The condition 1 above implies $s_i(0) = 0$.*

Definition 2 *A strategy profile S^* is a Markov perfect Nash equilibrium if it is admissible and the following condition are satisfied:*

For each initial condition $x \in X$ and every admissible strategy profile $S = (s_i, S^*_{-i})$,

$$\int_0^\infty u_i\{s_i^*[x^*(t)]\}e^{-\rho_i t}dt \geq \int_0^\infty u_i\{s_i[x(t)]\}e^{-\rho_i t}dt \quad \text{all } i \in N,$$

where $x^*(t)$ and $x(t)$ are trajectories starting from the common initial condition x, which are induced by strategy profiles S^* and S, respectively.

1.2 The result

[**Assumptions**]
(A.1) The natural growth function f is strictly concave and differentiable on $\mathrm{int}(X)$.
(A.2) $\lim_{x\to 0} f'(x) > \sum \rho_i$.
(A.3) For each $i \in N$, the felicity u_i is strictly increasing and differentiable on R_{++}.

Remark 2 *Under the assumptions (A.1) and (A.2), there exists the unique state $x > 0$ such that $f'(x) = \sum \rho_i$. Denote this state by x_c.*

Proposition 1 *Let a strategy profile S^* constitute a Markov perfect Nash equilibrium. Assume that the equilibrium has an interior stationary state x^*_{ss} such that $f(x^*_{ss}) - \sum s^*_i(x^*_{ss}) = 0$ and $x^*_{ss} > 0$. Also assume that the equilibrium strategy for each player is differentiable at $x = x^*_{ss}$. Then, under the assumptions (A.1), (A.2), and (A.3), the stationary state is: (1) asymptotically stable if $x^*_{ss} < x_c$, (2) unstable if $x^*_{ss} > x_c$.*

Proof. For each player $i \in N$, define the Hamiltonian as

$$H_i(c, x, \lambda_i) = u_i(c) + \lambda_i \left[f(x) - \sum_{j \neq i} s^*_j(x) - c \right].$$

Since $x^*_{ss} > 0$, $s^*_i(x^*_{ss})$ must be positive for all $i \in N$, otherwise player i can gain by deviating from the action $s^*_i(x^*_{ss})$. Therefore, at $x = x^*_{ss}$ the following equations are satisfied:

$$u'_i[s^*_i(x^*_{ss})] = \lambda_i \neq 0, \tag{1.1}$$

$$\dot{\lambda}_i/\lambda_i = 0 = \rho_i - f'(x^*_{ss}) + \sum_{j \neq i} s^{*\prime}_j(x^*_{ss}) \quad \text{each } i \in N. \tag{1.2}$$

Summing (1.2) over the players, we have

$$\sum \rho_i - N f'(x^*_{ss}) + (N-1) \sum s^{*\prime}_i(x^*_{ss}) = 0.$$

Therefore

$$f'(x^*_{ss}) - \sum s^{*\prime}_i(x^*_{ss}) = \frac{\sum \rho_i - f'(x^*_{ss})}{N-1}.$$

It is concluded that

$$x^*_{ss} \text{ is } \begin{cases} \text{asymptotically stable} & \text{if } x^*_{ss} \in (0, x_c), \\ \text{unstable} & \text{if } x^*_{ss} > x_c, \end{cases}$$

since f is strictly concave. ■

Immediate corollary is yielded for the case of symmetric equilibria.

Definition 3 *A Markov perfect Nash equilibrium is called symmetric if the equilibrium strategy of each player,* $s_i : X \to R_+$*, is common, that is,* $s_i(x) = s_j(x)$ *for all* $i, j \in N$ *and for all* $x \in X$*.*

Corollary 1 *Under the assumptions (A.1), (A.2), and (A.3), for any symmetric Markov perfect Nash equilibrium, the stationary states, if any, are at most countable.*

Proof. Assume that there exists a continuum of stationary states, $I_{ss} \subset X$. By the construction, $\mathrm{int}(I_{ss})$ has a positive measure and any $x \in \mathrm{int}(I_{ss})$ is neither asymptotically stable nor unstable. On the other hand, for any $x \in I_{ss}$, $s_i(x) = f(x)/N$. f is differentiable, so that $s_i(x)$ is differentiable on $\mathrm{int}(I_{ss})$. Therefore Proposition 1 is applicable to this case and we have $\mathrm{int}(I_{ss}) = \{x_c\}$, whose measure is zero. A contradiction appears. ■

Another immediate corollary is for the case where the natural growth function f has the carrying capacity.

Corollary 2 *Assume that (A.1), (A.2), and (A.3) as well as* $\lim_{x\to\infty} f(x) < 0$*. If a Markov perfect Nash equilibrium of the game* $(N, \{u_i, \rho_i\}_{i=1}^n, X, f)$ *is constituted by the strategies all of which are differentiable at the stationary states, then the equilibrium state trajectory from every initial state* $x \in X$ *converges to some stock level less than or equal to* x_c*.*

Proof. From (A.1),(A.2), and $\lim_{x\to\infty} f(x) < 0$, there is the unique $\bar{x} > x_c$ such that $f(\bar{x}) = 0$, *i.e.*, the carrying capacity of the resource. Obviously the equilibrium trajectory $x(t)$ is strictly decreasing if $x(t) > \bar{x}$. On the other hand, if there exists a stationary point on $(x_c, \bar{x}]$, it must be unstable by Proposition 1. Therefore the equilibrium trajectory starting from any $x \in \mathrm{int}(X)$ can not converge to the point on $(x_c, \bar{x}]$. ■

Remark 3 *If we drop the assumption (A.2) and, instead, adopt the assumption,* $\lim_{x\to 0} f'(x) \le \sum \rho_i$*, Proposition 1 is modified that every interior stationary state is unstable. Therefore Corollary 2 implies that if the resource has the properties that the growth rates are small enough (* $\lim_{x\to 0} f'(x) \le \sum \rho_i$*) and the carrying capacity exists, the equilibrium state trajectories necessarily converge to zero.*

Remark 4 *Consider a strategy profile* S *such that for each* $i \in N$*,*

$$s_i(x) = \begin{cases} s_i \ \textit{(constant)} & \textit{if } x > 0 \\ 0 & \textit{if } x = 0. \end{cases}$$

Corollary 2 implies that if S *constitutes a Markov perfect Nash equilibrium, then it must be the case of* $\sum s_i > \max\{f(x) | x \in X\}$*. That is, under the equilibrium strategy* S*, the resource becomes extinct within some finite time regardless of the initial stock level.*

The third corollary in this section is concerned with the efficient stationary state.

Definition 4 *The cooperative solution is the solution of the following problem:*

$$\max_{c_i(t)} \int_0^\infty \sum_{i=1}^n \{u_i[c_i(t)]e^{-\rho_i t}\}dt,$$

$$\textit{subject to } \dot{x} = f[x(t)] - \sum_{i=1}^n c_i(t),\ c_i(t) \geq 0,\ x(t) \geq 0, \textit{ for each } t \geq 0,$$

$$\textit{and } x(0) \in X \textit{ given.}$$

Corollary 3 *If a Markov perfect Nash strategy profile S^* of a resource game $(N, \{u_i, \rho_i\}_{i=1}^n, X, f)$ have an interior stationary state $x_{ss}^* > 0$ such that to keep x_{ss}^* is a cooperative solution and if each player's equilibrium strategy $s_i^*(x)$ is differentiable at $x = x_{ss}^*$, then the time discount rate of each player must be common, i.e., $\rho_i = \rho_j$ for all $i, j \in N$, and x_{ss}^* must be an unstable stationary state in the equilibrium.*

Proof. For the resource game $(N, \{u_i, \rho_i\}_{i=1}^n, X, f)$, pick up a Markov perfect Nash equilibrium strategy profile which has an interior stationary state $x_{ss}^* > 0$. Denote the equilibrium strategy for each player by $s_i^*(x)$. Then consider the cooperative problem associated with the game and starting from the stationary state:

$$\max \int_0^\infty \sum_{i=1}^n [u_i(c_i)e^{-\rho_i t}]dt,$$

$$\text{subject to } \dot{x} = f(x) - \sum_{i=1}^n c_i,\ x(0) = x_{ss}^* \text{ given.}$$

Define the Hamiltonian by

$$H(\{c_i\}_{i=1}^n, x, \lambda, t) = \int_0^\infty \sum_{i=1}^n [u_i(c_i)e^{-\rho_i t}] + \lambda[f(x) - \sum_{i=1}^n c_i].$$

When the optimal controls $c_i^*(t)$ $i \in N$ coincide with $s_i^*[x(t)]$, *i.e.*, $c_i^*(t) = s_i^*[x_{ss}^*]$ (positive constant), the following equations must be satisfied:

$$u_i'[s_i^*(x_{ss}^*)] = \lambda e^{\rho_i t} \quad \text{each } i \in N,$$
$$\dot{\lambda}/\lambda = -f'(x_{ss}^*).$$

Assume that s_i is differentiable at $x = x_{ss}^*$. Then compare these equations with (1.1) and (1.2). We have

$$-f'(x_{ss}^*) = \frac{\dot{\lambda}}{\lambda} = \frac{\dot{\lambda}_i}{\lambda_i} - \rho_i = -\rho_i = -f'(x_{ss}^*) + \sum_{j \neq i} s_j^{*\prime}(x_{ss}^*) \quad \text{each } i \in N.$$

Therefore this must be the case where $\rho_i = \rho_j = f'(x^*_{ss}) > 0$ for all $i, j \in N$. Also $\sum_{j \neq i} s^{*\prime}_j(x^*_{ss}) = 0$ must hold for each $i \in N$. This implies $\sum_{i=1}^N s^{*\prime}_i(x^*_{ss}) = 0$, so that the stationary state x^*_{ss} is unstable. ■

2 On the most rapid extinction when a resource has externality

Sorger (1998) proves the necessary and sufficient condition for the most rapid extinction to constitute a Markov perfect Nash equilibrium and provides some of the sufficient conditions. He used a model in which the resource has no externality. As a simple extension of Sorger's work, we consider in this section the case where the resource has externality.

2.1 Additional notation and assumptions

- The resource game is symmetric and described with $(N, u+v, \rho, X, f)$.
- The felicity takes a separably additive form, $u(c) + v(x)$, where c denotes consumption (or harvest) level of extracted resource flow and x denotes stock level of resource.

[Assumptions]
(A.4) $u : [0,k] \to R_+$ is strictly increasing, strictly concave, and differentiable on $(0,k]$. $u(0) = 0$ and $u(k) < \infty$.
(A.5) $v : R_+ \to R_+$ is strictly increasing, concave, and twice continuously differentiable on R_{++}. $v(0) = 0$ and $\lim_{x \to \infty} v(x) = \bar{v} < \infty$ exists.
(A.6) $\lim_{x \to 0} f'(x) = \infty$, $\lim_{x \to \infty} f'(x) < 0$, and $\max\{f(x) | x \in R_+\} < nk$.

Remark 5 *1. By (A.4), we assume that the capacity of harvest of each player is limited by k.*

2. The normalization of the felicity, $u(0) = 0, v(0) = 0$, in (A.4) and (A.5) is innoxious as long as we suppose that the felicity is bounded below.

Definition 5 *The following strategy is called most rapid extinction:*

$$s^x(x) = \begin{cases} k & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

- Denote by $T(x)$ the extinction time when all player adopt most rapid extinction and the initial stock is x.

- Denote by $V(x)$ the total payoff (life time utility) for each player when all player adopt most rapid extinction and the initial stock is x. Formally,

$$V(x) = \int_0^{T(x)} \{u(k) + v[(x(t)]\} e^{-\rho t} dt, \tag{2.1}$$

where $x(t)$ is the solution of $\dot{x}(t) = f[x(t)] - nk$ with the initial condition, $x(0) = x$.

2.2 Preliminaries

2.2.1 The extinction time

The extinction time $T(x)$ is implicitly expressed as

$$x(0) + \int_0^{T(x)} (f[x(t)] - nk)\, dt = 0, \tag{2.2}$$

where $x(t)$ is the solution of $\dot{x}(t) = f[x(t)] - nk$ with the initial condition, $x(0)$.

Choose an initial condition $x(0) = y \in \mathrm{int}(X)$ and the stock level $x \in (0, y)$ arbitrarily. Since the state trajectory is continuous and strictly decreasing, there is the unique time $\tau > 0$ such that $x(\tau) = x$. From (2.2), x and τ must satisfy the equation,

$$y - x + \int_0^{\tau} (f[x(t)] - nk)\, dt = 0.$$

Since $f[x(t)]$ is continuous in t, we can take the total differentiation with respect to x and τ. Hence we have

$$-dx + \{f[x(\tau)] - nk\} d\tau = 0.$$

Notice that the extinction time $T(x)$ satisfies $T(x) = T(y) - \tau$. Also notice that $y \in \mathrm{int}(X)$ and $x \in (0, y)$ are chosen arbitrarily. From these, we have the differential equation,

$$\frac{dT(x)}{dx} = -\frac{d\tau}{dx} = \frac{1}{nk - f(x)} \quad \text{all } x > 0.$$

Since the initial condition $T(0)$ is zero, the extinction time $T(x)$ is explicitly expressed as

$$T(x) = \int_0^x \frac{dy}{nk - f(y)}. \tag{2.3}$$

2.2.2 Total payoff function and the derivatives

Fix the initial stock $x \in \text{int}(X)$ and define $t(y) = T(x) - T(y)$ with domain $[0, x]$. Using the integration by substitution and the integration by parts, we have the explicit expression of the total payoff function $V(x)$:

$$\begin{aligned}
V(x) &= \int_0^{T(x)} \{u(k) + v[x(t)]\} e^{-\rho t} dt \\
&= \frac{1 - e^{-\rho T(x)}}{\rho} u(k) + \int_0^{T(x)} v[(x(t)] e^{-\rho t} dt \\
&= \frac{1 - e^{-\rho T(x)}}{\rho} u(k) + \int_x^0 v(y) e^{-\rho[T(x) - T(y)]} \left[-dT(y)/dy\right] dy \\
&= \frac{1 - e^{-\rho T(x)}}{\rho} u(k) + \frac{e^{-\rho T(x)}}{\rho} \int_0^x v(y) \left[de^{\rho T(y)}/dy\right] dy \\
&= \frac{1 - e^{-\rho T(x)}}{\rho} u(k) + \frac{e^{-\rho T(x)}}{\rho} \left\{ \left[v(y) e^{\rho T(y)}\right]_{y=0}^x - \int_0^x v'(y) e^{\rho T(y)} dy \right\} \\
&= \frac{1}{\rho} \left\{ [u(k) + v(x)] - e^{-\rho T(x)} \left[u(k) + \int_0^x v'(y) e^{\rho T(y)} dy\right] \right\}. \qquad (2.4)
\end{aligned}$$

This yields $V'(x)$ immediately:

$$\begin{aligned}
V'(x) &= \frac{1}{\rho} \left\{ v'(x) + e^{-\rho T(x)} \rho T'(x) \left[u(k) + \int_0^x v'(y) e^{\rho T(y)} dy\right] - v'(x) \right\} \\
&= \frac{e^{-\rho T(x)} \left[u(k) + \int_0^x v'(y) e^{\rho T(y)} dy\right]}{nk - f(x)} \qquad (2.5a) \\
&= \frac{[u(k) + v(x)] - \rho V(x)}{nk - f(x)} \quad \text{any } x \in \text{int}(X). \qquad (2.5b)
\end{aligned}$$

Also for $V''(x)$,

$$\begin{aligned}
V''(x) &= \frac{v'(x) - \rho V'(x)}{nk - f(x)} + \frac{f'(x)}{nk - f(x)} V'(x) \\
&= \frac{v'(x) - [\rho - f'(x)] V'(x)}{nk - f(x)} \quad \text{any } x \in \text{int}(X). \qquad (2.6)
\end{aligned}$$

2.2.3 The Weitzman's rule

We here show the relationship between the total payoff $V(x)$ and the temporal payoff $u(k) + v(x)$. Rearranging (2.5b) yields

$$\rho V(x) = u(k, x) + V'(x)\dot{x} \quad \text{all } x \in \text{int}(X), \qquad (2.7)$$

where $\dot{x} = f(x) - nk$.

This equation says that when all players adopt the most rapid extinction, the total payoff for each player is proportional to the temporal payoff plus the value of net investment (depreciation) evaluated by the marginal total payoff of the resource stock. We might call the right hand side of (2.7) net income in the broad sense. This relationship between total payoff and net income was found originally by Weitzman (1976) in the context of the competitive economy, so that the equation can be called the Weitzman's rule.

2.3 The result

Lemma 1 *For a symmetric resource game $(N, u+v, \rho, X, f)$ with the assumptions (A.1),(A.4)-(A.6), the most rapid extinction strategy profile S^x such that*

$$s_i^x(x) = s^x(x) = \begin{cases} k & \text{if } x > 0, \\ 0 & \text{if } x = 0 \end{cases} \quad \text{for all } i \in N$$

constitutes a symmetric Markov perfect Nash equilibrium if and only if

$$u'(k) \geq V'(x) \quad \text{all } x \in \mathrm{int}(X). \tag{2.8}$$

Proof. First, recall Remark 1 in this paper. Every admissible strategy profile S must satisfy $s_i(0) = 0$ for all $i \in N$. Therefore all we have to prove is just for positive resource stocks, *i.e.*, the case for $x \in \mathrm{int}(X)$. Second, notice that under the condition (2.8), the Weitzman's rule (2.7) turns to the Hamilton-Jacobi-Bellman's equation. That is, for all $x \in \mathrm{int}(X)$, the following equation holds:

$$\begin{aligned} \rho V(x) &= [u(k) + v(x)] + V'(x)[f(x) - ns^x(x)] \\ &= \max_{c \in [0,k]} [u(c) + v(x)] + V'(x)[f(x) - (n-1)s^x(x) - c]. \end{aligned} \tag{2.9}$$

(i) "only if" part. Assume the most rapid extinction strategy profile constitutes a Markov perfect Nash equilibrium. Then for each player $i \in N$, and for all $x \in \mathrm{int}(X)$, the Hamilton-Jacobi-Bellman's equation (2.9) must hold. This necessary condition is equivalent to the condition (2.8) as seen just above.

(ii) "if" part. Choose arbitrarily an admissible strategy profile S such that $S = (s_i(x), S_{-i}^x)$. Also pick up arbitrarily the initial stock $x_0 \in \mathrm{int}(X)$. Let $x(t)$ be the solution of $\dot{x}(t) = f[x(t)] - (n-1)s^x[x(t)] - s_i[x(t)]$ with the initial condition x_0 and let $s_i[x(t)] = c_i(t)$. Denote by $T \in (0, \infty]$ the time when the resource becomes extinct, *i.e.*, the first time satisfying $x(t) = 0$. Since the Hamilton-Jacobi-Bellman's equation (2.9) holds for all $x \in \mathrm{int}(X)$, we have along the state trajectory $x(t)$

$$\{\rho V[x(t)] - V'[x(t)]\dot{x}(t)\}e^{-\rho t} \geq \{u[c(t)] + v[x(t)]\}e^{-\rho t} \quad \text{if } t \in [0, T).$$

Integrating the both side over $[0, T]$ yields

$$-\int_0^T \frac{dV[x(t)]e^{-\rho t}}{dt} dt \geq \int_0^T \{u[c(t)] + v[x(t)]\}e^{-\rho t} dt.$$

The left hand side is solved as

$$V(x) - V[x(T)]e^{-\rho T} = V(x),$$

since if T is finite, $V[x(T)] = V(0) = 0$, and if T is infinite, $\lim_{t\to\infty} V[x(t)]e^{-\rho t} = 0$ because the trajectory is bounded above by the assumption (A.6). On the right hand side, notice that if $x(t) = 0$ then $c(t) = 0$, so that for $t \geq T$, $u[c(t)] + v[x(t)] = 0$. Hence $\int_0^T \{u[c(t)] + v[x(t)]\}e^{-\rho t}dt = \int_0^\infty \{u[c(t)] + v[x(t)]\}e^{-\rho t}dt$. Gathering the results on both sides, we get

$$V(x) \geq \int_0^\infty \{u[c(t)] + v[x(t)]\}e^{-\rho t}dt.$$

It is obvious that the equality holds if $s_i(x) = s^x(x)$. Therefore there are no other strategies which gain more than the most rapid extinction. That is, the most rapid extinction constitutes a Markov perfect Nash equilibrium if the inequality in (2.8) holds. ■

[**Additional notation**] Denote by x_ρ the stock level which satisfies $\rho - f'(x_\rho) = 0$.

Notice that by (A.6), there exists unique x_ρ for any $\rho \in R_+$. Also notice that if $x \in (0, x_\rho)$, $\rho - f'(x) < 0$, and if $x > x_\rho$, $\rho - f'(x) > 0$.

Lemma 2 *There exists the unique stock level $x^* > x_\rho$ such that $V'(x)$ is strictly increasing on $(0, x^*)$ and strictly decreasing on (x^*, ∞), i.e., $V'(x)$ is unimodal. At $x = x^*$, $V'(x^*) = v'(x^*)/[\rho - f'(x^*)]$ holds.*

Proof. Recall the expressions of $V'(x)$ and $V''(x)$ in (2.5a) and (2.6):

$$V'(x) = \frac{e^{-\rho T(x)}\left[u(k) + \int_0^x v'(y)e^{\rho T(y)}dy\right]}{nk - f(x)} \tag{2.5a}$$

$$= \frac{[u(k) + v(x)] - \rho V(x)}{nk - f(x)}, \tag{2.5b}$$

$$V''(x) = \frac{v'(x) - [\rho - f'(x)]\, V'(x)}{nk - f(x)}. \tag{2.6}$$

It follows from (2.5a) that $V'(x) > 0$ for all $x \in \text{int}(X)$, so that $\lim_{x\to\infty} V(x) \in (0, \infty]$ exists. Notice that for all $x \in \text{int}(X)$,

$$V(x) < \int_0^{T(x)} [u(k) + v(x)]e^{-\rho t}dt = \frac{1 - e^{-\rho T(x)}}{\rho}[u(k) + v(x)].$$

Hence

$$\lim_{x\to\infty} \rho V(x) \leq \lim_{x\to\infty} [1 - e^{-\rho T(x)}][u(k) + v(x)] = u(k) + \bar{v} < \infty.$$

Therefore as $x \to \infty$, the numerator of (2.5b) converges to zero, while the denominator, $nk - f(x)$, goes to ∞, so that $\lim_{x\to\infty} V'(x) = 0$. On the other hand, from (2.6) $V'(x)$ is strictly increasing on the interval $(0, x_\rho]$. Therefore there is some $\bar{x} \in (x_\rho, \infty)$ such that $V'(\bar{x}) \geq V'(x)$ for all $x \in \mathrm{int}(X)$. Such $\bar{x}$ must satisfy $V''(\bar{x}) = 0$, so that $[\rho - f'(\bar{x})]\, V'(\bar{x}) = v'(\bar{x})$ holds. However $d[\rho - f'(\bar{x})]V'(\bar{x})/dx = -f''(\bar{x})V'(\bar{x}) > 0$, while $v''(\bar{x}) \leq 0$. This implies that $\bar{x}$ is unique and we conclude $\bar{x} = x^*$. ■

Proposition 2 *For a symmetric resource game $(N, u + v, \rho, X, f)$ with the assumptions (A.1),(A.4)-(A.6),*

(1) the most rapid extinction constitutes a Markov perfect Nash equilibrium if the number of players n is sufficiently large, and,

(2) there is some $\bar{\rho} > 0$ such that the most rapid extinction constitutes a Markov perfect Nash equilibrium if and only if $\rho \in [\bar{\rho}, \infty)$.

Proof.
(1) From Lemma 1 and 2, it is sufficient that $u'(k) \geq V'(x^*)$ holds for the sufficiently large n.

Fix the initial stock $x \in \mathrm{int}(X)$. Since $\rho V(x) \in (0, u(k) + v(x))$ for any $x \in \mathrm{int}(X)$, it follows that

$$\lim_{n\to\infty} V'(x) = \lim_{n\to\infty} \frac{[u(k) + v(x)] - \rho V(x)}{nk - f(x)} = 0.$$

In particular, $V'(x^*) \to 0$ as $n \to \infty$, including the case that $x^* \to \infty$ as $n \to \infty$. This implies that there exists $\bar{n} > 1$ such that if $n \geq \bar{n}$ then $u'(k) \geq V'(x^*)$
(2) First, recall the explicit expression of $V(x)$, (2.4).

$$V(x) = \frac{1}{\rho}\left\{[u(k) + v(x)] - e^{-\rho T(x)}\left[u(k) + \int_0^x v'(y)e^{\rho T(y)}dy\right]\right\}. \tag{2.4}$$

Then fix the initial stock $x \in \mathrm{int}(X)$ and take $\lim_{\rho\to\infty} \rho V(x)$:

$$\begin{aligned}
&\lim_{\rho\to\infty} \rho V(x) \\
&= [u(k) + v(x)] - \lim_{\rho\to\infty}\left[u(k)e^{-\rho T(x)} + \int_0^x v'(y)e^{-\rho[T(x)-T(y)]}dy\right] \\
&= u(k) + v(x).
\end{aligned}$$

From this, it follows that

$$\lim_{\rho\to\infty} V'(x) = \lim_{\rho\to\infty} \frac{[u(k) + v(x)] - \rho V(x)}{nk - f(x)} = 0.$$

In particular, $V'(x^*) \to 0$ as $\rho \to \infty$, including the case that $x^* \to \infty$ as $\rho \to \infty$.

Next, recall (2.5a) and calculate $\partial V'(x)/\partial\rho$:

$$\begin{aligned}\frac{\partial V'(x)}{\partial\rho} &= \frac{\partial}{\partial\rho}\left(\frac{e^{-\rho T(x)}\left[u(k)+\int_0^x v'(y)e^{\rho T(y)}dy\right]}{nk-f(x)}\right)\\ &= -T(x)V'(x)+\frac{e^{-\rho T(x)}\int_0^x T(y)v'(y)e^{\rho T(y)}dy}{nk-f(x)}\\ &= \frac{-e^{-\rho T(x)}\left[T(x)u(k)+\int_0^x[T(x)-T(y)]v'(y)e^{\rho T(y)}dy\right]}{nk-f(x)}\\ &< 0.\end{aligned}$$

Therefore, invoking the envelop theorem, we have $dV'(x^*)/d\rho = \partial V'(x^*)/\partial\rho < 0$. Furthermore, for every $x \in \mathrm{int}(X)$, as $\rho \to 0$, $V'(x)$ goes to ∞ since $\partial V'(x)/\partial\rho < -e^{-\rho T(x)}T'(x)T(x)u(k)$. From these observations, we can conclude that there exists some $\bar{\rho} > 0$ such that $u'(k) \geq V'(x^*)$ if and only if $\rho \geq \bar{\rho}$. ∎

Remark 6 *In this section, we choose R_+ as the state space X. Since the natural growth function f is assumed to have the carrying capacity, it might be natural to restrict the state space to the interval in which the growth rates of the resource are non negative. However, it is easily verified that this restriction does not require any changes to above results.*

3 A remedy for implementation of the cooperative solution

In this section, we show how a remedy can implement the cooperative solution defined as:

$$\max_{c_i(t)} \int_0^\infty \sum_{i=1}^n \{u_i[c_i(t)]e^{-\rho_i t}\}dt,$$

$$\text{subject to } \dot{x} = f[x(t)] - \sum_{i=1}^n c_i(t),\ c_i(t) \geq 0,\ x(t) \geq 0, \text{ for each } t \geq 0,$$

$$\text{and } x(0) \in X \text{ given.}$$

We use the following assumptions:

[Assumptions]
(A.7) The natural growth function $f : R_+ \to R$ is concave and C^1.

(A.8) $u_i : R_+ \to \bar{R}$ is a strictly concave and strictly increasing C^1 function for each $i \in N$.
(A.9) There exists a solution of the cooperative problem $(x^*(t), c_i^*(t))$ such that $c_i^*(t)$ is piecewise continuous and $(x^*(t), c_i^*(t)) > 0$ for all $t \in R_+$.

Remark 7 *1. By adopting (A.7) instead of (A.1), we include the case where the resource is exhaustible as well as the case of unbounded growth as AK models.*

2. It is well known that under the assumptions (A.7) and (A.8), the solution $(x^*(t), c_i^*(t))$ is unique if it exists.

3. The existence of the cooperative solution in (A.9) is a premise to consider the implementation of the cooperative solution. On the other hand, the interiority of the solution and the piecewise continuity of the optimal control are technical assumptions for Pontryagin's maximum principle including the transversality condition to be necessary for the cooperative solution.

Lemma 3 *There exists a non-negative absolutely continuous function $p : R_+ \to R_+$, and the cooperative solution satisfies the following conditions:*

$$\sum_{i \in N} u_i'[c_i^*(t)]e^{-\rho_i t} - p(t) = 0, \quad \textit{all } t \in R_+, \tag{3.1}$$

$$\dot{p}(t) = -p(t)f'[x^*(t)], \quad \textit{a.e. } t \in R_+, \tag{3.2}$$

$$\lim_{t \to \infty} p(t)x^*(t) = 0. \tag{3.3}$$

Proof. See Kamihigashi(2000). ∎

Remark 8 *It is also well known that (3.1),(3.2), and (3.3) is the sufficient condition for the cooperative solution.*

The idea for implementation of the cooperative solution is to establish some markets by introducing "money". We postulate the presence of the benevolent and forcible government. The government takes from each player the right to freely use the resource at the initial time, and as the compensation for it, the players are given some amount of money at the same time. All players are prohibited from harvesting resources without paying some harvest fee to the government. For simplicity, let the unit of money be one of the resource harvest, *i.e.*, to harvest h units needs h units money. The money can be used as the device to store the wealth of each player. For the money in order to function as wealth, the government guarantees that the money can be exchanged for same amount of harvests at any time once the resource should become extinct.

Also the government announces that the money as wealth bears interest. The interest rates are time dependent but predetermined at the initial time.

By this remedy, the players yield up their right of free use of the common property resource but they get their private wealth and a kind of capital market. The money can also be used by the purchase of harvests, so that the remedy also will found a market of harvests.

To state the remedy formally, we add the following notation.

- $M_i \geq 0$ the amount of money given to player i at the initial time $t = 0$.
- $r(t)$ the interest rates at $t \in R_+$.
- $M_i(t)$ the amount of money held by player i at $t \in R_+$.
- $h_i(t) \in R_+$ the harvest rates of player i at $t \in R_+$.
- $c_i(t) \in R_+$ the consumption rates of player i at $t \in R_+$.
- $q(t) \geq 0$ the market price of harvests at $t \in R_+$.
- $\mathcal{R} = (r(t), \{M_i\}_{i \in N})$ the remedy.

[Additional Assumption]
(A.10) The interest rates $r(t)$ is a measurable function on R_+.

When the government adopts a remedy $\mathcal{R}$, the players get out of the common resource game or the "natural state", and come to play the another game. In the game with $\mathcal{R}$, each player $i \in N$ solves the following optimization problem:

$$\max_{c_i(t), h_i(t)} \int_0^\infty u_i[c_i(t)] e^{-\rho_i t} dt, \tag{3.4}$$
$$\text{subject to } \dot{M}_i(t) = r(t)M_i(t) - h_i(t) + q(t)[h_i(t) - c_i(t)] \quad \text{a.e. } t \in R_+,$$
$$c_i(t), h_i(t), M_i(t) \geq 0 \quad \text{all } t \in R_+, \quad M_i(0) = M_i \text{ given.}$$

Remark 9 *$M_i(t) \geq 0$ in the constraint is a non Ponzi game condition. Any player has no income other than the interest revenues, so that once they have a debt, they cannot leave debt forever.*

It is obvious, but important to see that in the problem (3.4), there appears no strategic interaction among players. It is because a remedy $\mathcal{R}$ breaks off the relations between the resource and the well-being of each player. The problem is completely individual as long as the players have a belief that their money can be exchanged for the consumption good even if the resource should become extinct. The government announces to guarantee the exchange. However one might doubt the word. Whether the announcement is credible is a practical question and depends on the case to which we would apply this remedy in the real world. Later we shall discuss the credibility. But before that, let us examine how the remedy could work to implement the cooperative solution.

Lemma 4 *For any remedy $\mathcal{R}$, $q(t) = 1$ for all $t \in R_+$.*

Proof. Suppose $q(t) > 1$. Then every player chooses to harvest and will not purchase the harvest in market. Suppose $q(t) < 1$. Then every player chooses to purchase harvests in market, so that no one will supply the harvests. ■

Using this lemma, we can rewrite the optimization problem (3.4) as

$$\max_{c_i(t),h_i(t)} \int_0^\infty u_i[c_i(t)]e^{-\rho_i t}dt, \tag{3.5}$$
$$\text{subject to } \dot{M}_i(t) = r(t)M_i(t) - c_i(t) \quad \text{a.e. } t \in R_+,$$
$$c_i(t), M_i(t) \geq 0 \quad \text{all } t \in R_+, \quad M_i(0) = M_i \text{ given.}$$

Remark 10 *The lemma cannot be valid if the credibility of the money should be lost, i.e., if there were a player who thought that the money would have no value once the resource should become extinct.*

Proposition 3 *Let $x \in \operatorname{int}(X)$ be the initial stock of the resource and let $(x^*(t), \{c_i^*(t)\}_{i\in N})$ the cooperative solution starting from x. Then under a remedy $\mathcal{R}$ all players choose the cooperative solution if and only if $\mathcal{R} = (r(t;x), M_i(x))$ such that*

$$r(t;x) = f'[x^*(t)], \tag{3.6}$$
$$M_i(x) = \int_0^\infty c_i^*(t)e^{-\int_0^t r(s)ds}dt. \tag{3.7}$$

Proof. Fix player $i \in N$. Let $\lambda_i(t)$ be a non-negative absolutely continuous function in $t \in R_+$ and define the present value Hamiltonian with

$$H_i[c, M, \lambda(t), t] \equiv u_i(c)e^{-\rho_i t} + \lambda_i(t)[r(t)M - c], \quad \text{each } t \in R_+.$$

If there exists $(c_i^e(t), M_i^e(t), \lambda_i(t))$ such that

$$H_i[c_i^e(t), M_i^e(t), \lambda_i(t), t] = \max_{c\geq 0} H_i[c, M_i(t), \lambda_i(t), t] \quad \text{a.e. } t \in R_+, \tag{3.8}$$
$$\dot{\lambda}_i(t) = -\lambda_i(t)r(t) \quad \text{a.e. } t \in R_+, \tag{3.9}$$
$$\lim_{t\to\infty} \lambda_i(t)M_i^e(t) = 0, \tag{3.10}$$
$$M_i^e(0) = M_i(x), \quad \dot{M}_i^e(t) = r(t)M_i^e(t) - c_i^e(t) \quad \text{a.e. } t \in R_+, \tag{3.11}$$
$$\text{and } c_i^e(t), M_i^e(t) \geq 0 \quad \text{each } t \in R_+, \tag{3.12}$$

then $(c_i^e(t), M_i^e(t), \lambda_i(t))$ is the unique solution of the problem (3.5).

Set $c_i^e(t) = c_i^*(t)$. Immediately, from (3.1), we can see that (3.8) holds if and only if $\lambda_i(t) = p(t)$, so that we set $\lambda_i(t) = p(t)$. Then we can see from (3.2) that the adjoint equation (3.9) holds if and only if $r(t) = f'[x^*(t)]$, so that the condition (3.6) is necessary for the remedy to work. Now we turns to the transversality condition, (3.10). Solve the linear differential equation, (3.11). It is rewritten as

$$\frac{d\left[M_i^e(t)e^{-\int_0^t r(s)ds}\right]}{dt} = -c_i^e(t)e^{-\int_0^t r(s)ds},$$

so that we have

$$\begin{aligned} M_i^e(t) &= e^{\int_0^t r(s)ds}\left[M_i^e(0) - \int_0^t c_i^e(\tau)e^{-\int_0^\tau r(s)ds}d\tau\right] \\ &= e^{\int_0^t r(s)ds}\left[M_i(x) - \int_0^t c_i^*(\tau)e^{-\int_0^\tau r(s)ds}d\tau\right]. \end{aligned}$$

On the other hand solving (3.9) yields

$$\lambda_i(t) = \lambda_i(0)e^{-\int_0^t r(s)ds} = p(0)e^{-\int_0^t r(s)ds}.$$

Therefore

$$\lambda_i(t)M_i^e(t) = p(0)\left[M_i(x) - \int_0^t c_i^*(\tau)e^{-\int_0^\tau r(s)ds}d\tau\right].$$

We can see that (3.10) holds, *i.e.,* $\lambda_i(t)M_i^e(t) \to 0$ as $t \to \infty$ if and only if the equation (3.7) holds. Finally, it is easily verified that non-negativity constraints, (3.12), are satisfied. For non-negativity of $M_i^e(t)$, it is seen from

$$\begin{aligned} M_i^e(t) &= M_i(x) - \int_0^t c_i^*(\tau)e^{-\int_0^\tau r(s)ds}d\tau \\ &= \int_0^\infty c_i^*(t)e^{-\int_0^t r(s)ds}dt - \int_0^t c_i^*(\tau)e^{-\int_0^\tau r(s)ds}d\tau \\ &= \int_t^\infty c_i^*(t)e^{-\int_0^t r(s)ds}dt \geq 0. \end{aligned}$$

For non-negativity of $c_i^e(t)$, it is trivial since $c_i^e(t) = c_i^*(t) > 0$. ■

[Note on the remedy R**]**

1. While the original resource game might have continuum of Nash equilibria in general, the equilibrium for the game with the remedy $\mathcal{R} = (r(t;x), M_i(x))$ is unique. Therefore the remedy can certainly implement the cooperative solution. This is an advantage that the remedy has. The Pigovian tax[1], for example, might not have this certainty property. The

[1] Clemhout and Wan (1985) suggests use of the Pigovian tax under the premise that players adopt linear strategies.

tax rate is the difference between two shadow prices of the resource stock, one supporting the efficient path and another supporting an equilibrium path. Therefore the effective tax rates (trajectory) depends on which equilibrium is played. However there are uncountably many equilibria. Hence we could not determine the appropriate tax rates nor be assured that the tax rate can correct inefficiency.

2. Although at the initial time the government takes from players their rights of free resource use, instead of that they are given individual wealth which can be utilized freely. On the other hand, the command control regime deprive people of both their property rights of the resource and the freedom of their choice.

3. Also from the viewpoints of individual rationality, the remedy has a desirable property. Notice that the problem (3.4) can include weights for each player's felicity. This implies that by choosing appropriate weights, through the remedy we can implement a cooperative solution in which every player could become better off than status quo of the "natural state". If the players are rational, they are willing to accept such a remedy.

4. As more practical argument, open access resource problems are often observed in the rural areas in developing countries. The desertification and the deforestation there include the problem without exception. For many cases, the people suffering from the "tragedy of commons" are extremely poor and the society is possessed with inequality. Related the extreme poverty, Dasgupta (1982) argues how we could recommend to impose the people barely surviving on tax. In this point, the remedy has an advantage since it does not deprive players unilaterally. As for the inequality of rural societies, Dasgupta and Mäler (1997) shows with several empirical studies that privatization of common property resources could lead the poorer among people in the poor area into more misery because of the inequitable division of the common property resource. Privatization is a solution to attain efficient resource uses if the resource could be divided into individual ownerships. However the equity problem remains. Also for this point, the remedy has a desirable property. By choosing appropriate weights for each player's felicity and adopting the corresponding remedy, it is possible to implement the efficient and equitable resource use whatever the equity is. This property is a variation of the second theorem of welfare economics.

5. Although the remedy has some advantages as mentioned so far, it has a flaw. For each player, the strategy is not only time consistent, but subgame perfect by the construction of the problem (3.4). However the strategy of the government is just time consistent, not subgame perfect. On the equilibrium path, once some player takes a wrong action, since then the state trajectory has become different from one that the government expected, *i.e.*, the collective outcome has become inefficient. But let us

point out that the flaw is common to many models based on the rational expectation hypothesis.

[On the credibility of the money]

Now we argue the credibility of the money when the resource should become extinct. First we show two examples that the money is credible.

- The case of fuel woods

  The case is typically observed in the desertification in rural areas in developing countries located in arid area, which has been one of the global environmental issues

  The government could easily guarantee credibility of the money. If the forest, the common property resource under consideration, should become extinct completely, then the government would only have to purchase fuel woods from somewhere.

- The case of pollution substances

  Reinterpret the resource stock as an environment index, the natural growth function f as the natural assimilation function, and the harvest rates h as the emission rates of pollution. The model in this section is applicable to a pollution or environmental problem when we consider a problem which has a critical threshold $x = 0$ such that there is no externality $x \geq 0$, but if it should fall in negative value, the society inevitably goes to ruin owing to irreversible and cumulative deterioration of the environment.

  In this case, the money could be still credible if the government has a technology to eliminate the emitted pollutions. Such a technology might be an end-pipe one, a purification technology, or a mix of both, depending on the problem under consideration.

The examples tell us when the money is credible. One of the typical cases is that the resource flow can be completely substituted for another good, which can be freely purchased in some market by the government. The other is that the resource stock can be augmented by the government in spite of the extraction. Formally both conditions are same. However from a practical view, the latter requirement would be more difficult for many cases.

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